HWS-9718, NIKHEF-96-035, hep-th/9707147

Anyon Statistics and the Witten Index

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Using the theory of supersymmetric anyons, I extend the definition of the Witten index to $2+1$ dimensions so as to accommodate the existence of anyon spin and statistics. I then demonstrate that, although in general the index receives irrational and complex contributions from anyonic states, the overall index is always integral, and I consider some of the implications and interpretations of this result.

6/97; revised 11/97

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1. Introduction

The Witten index [1] has proven to be a useful tool for understanding the dynamics of supersymmetric theories, as well as for finding deep and illuminating connections between physics and a variety of areas of mathematics. The conventional informal definition of the Witten index is $\mathrm{tr}(-1)^F$, which can be regulated into the more rigorously defined object, $\mathrm{tr}(-1)^F e^{-\beta H}$. Because of the degeneracy between bosonic and fermionic states in supersymmetric theories, the states of non-zero energy give a net zero contribution to the Witten index, and so the Witten index is equal to the difference between the number of bosonic and the number of fermionic zero energy states, which is an integer. (In a theory with a continuous spectrum, a fractional value is possible, due to subtleties in the cancellations between densities of states [2].)

In $2+1$ dimensions, however, states of any real spin and statistics can arise [3]. A standard way of obtaining exotic spin and statistics is to couple the fields to a gauge field with Chern-Simons term. What becomes of the index in this context? This paper examines this question. We will first define the Witten index in a way that incorporates anyons, obtaining an expression that generically involves complex and irrational terms. In the subsequent section, we show that, nonetheless, regardless of the presence of Chern-Simons terms and anyons, the index is an integer. This result then motivates the definition of an alternative index, using a grading based on whether fields satisfy canonical commutation or anticommutation relations. This alternative index is not as fundamental, but it is explicitly a sum of integers, and it necessarily has the same value as the fundamental anyon index in any quantized field theory, and so helps us understand the integral nature of the fundamental Witten index in theories with anyons.

That the Witten index is integer-valued even in anyonic theories is, perhaps, a disappointing result, as exotic values for the Witten index would be most interesting; on the other hand, the fact that whatever states with exotic spin and statistics do arise must do so in way that will produce an integral index is a potentially informative constraint on the behavior of anyonic field theories, and is worthy of careful consideration in particular models.

Before continuing, the reader should note that this paper is about the effects of Chern-Simons terms on the Witten index, and so I do not discuss those issues which may be relevant to the index but that are not pertinent to the anyonic case.

2. Statistics and the Witten Index

In $2+1$ dimensions, as is well-known, anyon statistics are possible. The standard definition of statistics in $2+1$ dimensions amounts to associating a phase of $\exp(2\pi ij)$ under the interchange of two objects of spin j. Since the rotation group in $2+1$ dimensional theories allows states of any real spin, the statistics in $2+1$ dimensions can yield any complex phase under particle interchange.

Fundamentally, this definition of statistics is made with reference to the topological properties of the paths of particles in configuration space; anyon statistics arise from the general assignment of phases to the topologically distinct sectors of multi-particle configuration space. In the framework of the braid group, anyons correspond to one-dimensional representations of the braid group.[1]

Now in supersymmetric theories with anyons, supersymmetry pairs finite energy states of spin j and $j+\frac{1}{2}$ [4]. Of course, states of zero energy are still supersymmetry singlets. Consequently, the natural definition of the Witten index I_A for a supersymmetric theory in $2+1$ dimensions is

$$I_A = \mathrm{tr}\big(e^{2\pi iJ}e^{-\beta H}\big) \ , \tag{2.1}$$

where J is the generator of spatial rotations, and thus $e^{2\pi iJ}$ generalizes $(-1)^F$. Because of the pairing of states provided by supersymmetry, I_A is a topological index, with all the standard topological properties that we associate with the Witten index in more familiar contexts. The index I_A only receives contributions from zero energy states. If I_A is non-zero, supersymmetry is not broken. The index I_A is unchanged by supersymmetric deformations of the parameters (provided, as usual, that said deformations do not radically alter the Hilbert space of the theory — see below), and can be reliably and exactly calculated in any approximation scheme that respects supersymmetry. In particular, neither the addition of massive particles nor the deformation of the parameters of the theory in a way that does not change the asymptotic behavior of the field theory potential can alter the value of the index.

[1] One can also construct non-abelian statistics based on higher dimensional representations of the braid group, using, for example, couplings to non-abelian gauge fields with Chern-Simons terms. The methods and results of this paper that show that the index is integral in ordinary anyon theories are equally valid in the non-abelian case. For simplicity, we will use ordinary anyons if a particular example is needed.

Let us consider the application of this index to a simple model. This example does not deal directly with anyonic features, but serves as a preview of the argument in the next section. The model is given by the Lagrangian

$$\mathcal{L} = \int d^2\theta \big(\frac{1}{2} D^\alpha \Phi D_\alpha \Phi + \frac{g}{3}\Phi^3 - \lambda\Phi \big) \ , \tag{2.2}$$

where Φ is a real superfield, whose physical components are a real scalar ϕ and a two-component spinor ψ. In terms of the physical component fields, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{i}{2}\bar{\psi}\gamma^\mu\partial_\mu\psi + \big(g\phi^2 - \lambda\big)^2 + 2g\phi\bar{\psi}\psi \ . \tag{2.3}$$

Without loss of generality, we take $g > 0$. The constant λ is real.

This theory has two phases: one in which its Z_2 symmetry is manifest, when $\lambda < 0$; and one in which the Z_2 is broken, when $\lambda > 0$. Since the index I_A is unchanged by changes in λ, we calculate I_A in the easier phase. For $\lambda < 0$, there are no classical zero energy configurations, and so the index is zero. In the Z_2-broken phase, one still has $I_A = 0$ There are now two classical minima with zero energy,

$$\phi = \pm\sqrt{\frac{\lambda}{g}} \ . \tag{2.4}$$

The excitations about each of these minima are all massive, and so $I_A = 0$ implies

$$e^{2\pi i j_1} + e^{2\pi i j_2} = 0 \ , \tag{2.5}$$

where j_1 and j_2 are the spins of the states associated with the two minima. Thus these two classical vacua correspond to two quantum states that have spins that differ by a half-integer. (If there were no possibility of anyons in $2+1$ dimensions, one could conclude on the basis of index arguments alone that one of these states is fermionic and the other bosonic.)

This is as far as the index alone can takes us. One can gain further insight into the theory by combining index-based results with other pieces of information about the theory (e.g., requiring that the ground state leave rotational invariance unbroken, or invoking the connection between classical vacua and kinks [5] [6]). We do not delve into these possibilities here.

Equating the index in distinct phases is a powerful tool, and we now use such a method to analyze the index in supersymmetric anyon theories.

3. The Anyonic Index is Integral

In this section, we will show that the Witten index in anyonic theories is integral, despite anyon effects.[2] We will do this by formulating the anyons using a Chern-Simons term, and then by invoking the topological invariance of the Witten index to find a regime in which, despite the presence of a Chern-Simons term, the index can be demonstrated easily to be integral. This result holds whether the gauge fields in the Chern-Simons term are auxiliary or dynamical.

Our plan is as follows. We first consider an apparently simple argument that it is implausible for the index to be non-integral, but discover that it is too naive, and indeed is unsound and logically inconsistent. Understanding the flaws in this argument, however, enables us then to develop a valid argument that the index remains integral even in the presence of Cherns-Simons terms.

Consider, therefore, the case that space is given by R^2. How might the index depend on the Chern-Simons coefficient in this case? On R^2, the (abelian) Chern-Simons coefficient is not quantized, and so we may apparently vary it without varying the index. As this coefficient is varied, the spins of the fields, and in turn the statistics of the particles they create, also vary continuously. For the index to remain unchanged when this occurs, it seems plausible that only those states whose statistics are unaffected by the change in this coefficient would make a net contribution to the index (i.e., the gauge neutral states), while the states with non-zero gauge charge would cancel each other. The gauge neutral states, in turn, all make integral contributions to the index, and so it would be natural to conjecture that the Witten index is integral in anyonic theories, as it seems implausible that the index could be fractional and yet remain unchanged as the Chern-Simons coefficient varied. (Indeed, if the gauge charges of the matter fields in a $U(1)$ gauge theory are commensurate, one can vary the Chern-Simons coefficient to a point where all the perturbative states have an integer or half-integer spin, thereby producing an integer index.)

Unfortunately, this argument, even as a plausibility argument, is flawed. Changing the Chern-Simons coefficient on R^2 changes the theory discontinuously; infinitesimal changes in the coefficient can change whether the wavefunctions live on a finite or infinite cover of the plane, for example. Furthermore, in order to define the theory properly, indeed to

[2] Whether it might be non-integral due to other effects such as those discussed in [2] is not at issue here.

calculate quantities such as the Witten index, one typically formulates the theory with space having the topology of a torus or other Riemann surface. However, on such a surface, the Chern-Simons coefficient is quantized, and therefore cannot be continuously varied. (For a detailed treatment of the constraints on the appearance of exotic statistics on Riemann surfaces, see [7].) Thus in order to address our problem, we need an approach that does *not* depend on varying the value of the Chern-Simons coefficient, and ideally we would like an approach that does not depend on the choice of spatial manifold. As a bonus, it would be nice if our method applied to both abelian and non-abelian Chern-Simons terms. Fortunately, a method that meets all three of these criteria exists.

Our goal, then, is to demonstrate that, in the presence of a Chern-Simons term, the index I_A, even though it generically receives non-integral contributions, is integral.[3] This does not rule out states of fractional spin and statistics, of course; rather, it is that the net contribution of such anyonic states is nonetheless integral.

The argument is as follows. Consider the index I_A in a theory in which a $U(1)$ or non-abelian Chern-Simons term endows the fields with anyon spin and statistics. We may add a massive superfield to this theory without altering the index. We therefore add a massive field which has a $U(1)$ or non-abelian gauge charge and a quartic (or hexadic, since this is $2+1$ dimensions) scalar potential.

Because the behavior of the potential at large field strength will be dominated by the quartic or hexadic terms, changing the coefficient of the quadratic term will not change the index [1]. Consequently, we vary this coefficient in the potential until it is negative, thus producing the Higgs mechanism. In the Higgs phase, anyon behavior is *not* induced in the perturbative spectrum, as observed in [8] and discussed in detail in [4]. The interested reader should consult these references, but the essential ideas are easy to summarize here. In the Higgs phase, there is no Aharonov-Bohm effect associated with electromagnetic point charges; this is because the solutions to the equations of motion for the combination $B + \mu A^0$ (where B is the magnetic field, μ the Chern-Simons mass, and A^0 the temporal component of the gauge field) are now exponentially damped, with a characteristic distance given by the inverse of the Higgs mass. In addition, the gauge field propagator is short range, and so there are no anomalous spin contributions generated in perturbation theory. Consequently the fields of the theory retain their canonical spin and statistics, and the

[3] That is, integral despite anyon effects. A mechanism such as that of [2] may still occur, as in any number of dimensions. Here, we are only interested in the particular effects of anyons.

perturbative spectrum therefore consists of states of integral and half-integral spin. Thus, in the Higgs phase, the index I_A is necessarily integer-valued. But the index has the same value in the Higgs (broken) phase as in the anyon (unbroken) phase [1], and so the original theory, although it contains anyons, must have integral index.

In preparation for the next section, we note that since the Higgs field has no direct coupling to the other matter fields, the only effect of the non-zero Higgs expectation value on the perturbative calculation of the index is to modify the gauge multiplet masses and to cause the charged fields to have the canonical statistics. Thus the only difference between the Higgs and anyon phase calculations of the index is the difference in the statistics. This shows us, then, that calculating the index using the naive statistics rather than the correct anyon statistics will yield the correct value of the index in the anyon phase. We return to this point in the next section.

These results represent a significant constraint on the behavior of supersymmetric theories in $2 + 1$ dimensions. To conclude our discussion of these results, it is worth making two additional observations.

First, it is known that in the Higgs phase of abelian Chern-Simons models, there may be non-perturbative states, such as vortices, with anyon spin. Since perturbative index calculations are exact, the net contribution of such non-perturbative states to the index must be zero, and so such states cannot change our conclusion. Indeed, the vortices are massive, and so obviously give zero contribution to the index, as expected.

Second, note that the above argument that I_A is integral clearly applies as well to the case that the exotic particle statistics arise from a non-abelian Chern-Simons term. Thus our finding is not restricted to the simple case of anyons, but encompasses their non-abelian generalization as well.

4. An Alternative Index

The results of the preceding section, which showed that the index is integral despite non-integral contributions in the sum, suggest the possibility of defining an alternative index that is automatically an integer but that has the same value as the Witten index I_A defined above. To this end, we introduce a notion of statistics for quantized field theories based on how fields are quantized. Let us define an operator Ω with the properties that $\Omega^2 = 1$ and that Ω (anti)commutes with those fields that are quantized with canonical (anti)commutation relations. In addition, we specify that the vacuum have eigenvalue $+1$

under the action of Ω; without this, we have only specified the relative value of Ω on the states in the Hilbert space. We use this operator to give an alternative definition of statistics in $2+1$ dimensional quantized field theory. The operator Ω has eigenvalues $+1$ and -1, and thus provides a twofold grading of the fields, of the creation and annihilation operators associated with the fields, and hence of the Fock space as well. Since one can formulate a supersymmetric theory by quantizing superfields in superspace, and since a superfield can be expanded in terms of a Grassman parameter θ, each superfield, obviously, pairs canonically commuting and anticommuting fields, as therefore does the supercharge. Therefore Ω anticommutes with the supersymmetry charge, and supersymmetry pairs Fock space states of $\Omega=+1$ with Fock space states of $\Omega=-1$.

Since supersymmetry pairs states of $\Omega=+1$ with states of $\Omega=-1$, we are led to define the alternative index

$$I_Z = \mathrm{tr}\Big(e^{-\beta H}\Omega\Big) \ . \tag{4.1}$$

Since $\Omega^2 = 1$, the index I_Z is manifestly integral. Since Ω anticommutes with the supersymmetry generators, the index I_Z also possesses all the expected properties of a Witten index: it can receive net non-zero contribution only from the zero energy states; when it is non-zero, supersymmetry is unbroken; it is unchanged by supersymmetric deformations of the parameters of a theory; and it is reliably and exactly calculated in supersymmetric approximation schemes. Note also that I_Z is the naive index that would be calculated by someone analyzing a field theory with a Chern-Simons gauge field mass term who did not know that the Chern-Simons term generates anyon statistics for charged fields.

In higher dimensions, the two indices I_A and I_Z are manifestly identical to each other, term by term in the sums, due to the spin-statistics theorem. Here, however, the indices I_A and I_Z are in principle qualitatively distinct from each other. One way of understanding the difference between these two indices is that I_Z merely keeps track of the pairing of states into superpartnerships, whereas I_A also keeps track of the spin associated with each superpartner pairing. Thus it is that I_Z is a sum of integers, while I_A is not, which was the reason for defining I_Z in the first place. It is important, however, to recognize that the quantity I_Z is defined in a less general context than I_A is, since I_Z makes explicit reference to the fields of a theory and to the quantization prescription, whereas I_A only makes reference to the Hilbert space of states and two of the operators (J and H) that act on that space.

Thus, by introducing an alternate definition of statistics, we have now identified two distinct Witten indices for $2+1$ dimensional theories. Nonetheless, these two indices are always equal in value. Recall our calculation of the index I_A in the previous section, in which we included a Higgs field which manifestly did not alter the index in the unbroken phase, and which enabled us to relate the index in two different phases of the theory. This same technique is the basis for establishing the equality of I_Z and I_A. Since in any Chern-Simons theory, the naive statistics are equivalent to the statistics defined via the operator Ω introduced above, in the Higgs phase, where anyon spin and statistics are not generated for the fields of a theory, the two indices I_A and I_Z are automatically equal to each other. In addition, I_Z, like I_A, is unchanged by changes in the Higgs field mass parameter, and thus each index independently has the same value in the Higgs phase as it does in the anyon phase. Therefore, we see that I_A and I_Z are equal in the anyonic phase of the theory with the added Higgs superfield, and consequently we see that $I_A = I_Z$ in the original theory, too, since the added Higgs superfield does not change the value of either index. And since I_Z is automatically an integer, we see again by this approach that I_A must be integral, too.

5. Concluding Thoughts: Review, Discussion, and Speculation

Using the equivalence of the index in Higgs and anyon phases, we have demonstrated that the Witten index is always integral in supersymmetric anyon theories. The arguments apply to theories coupled to abelian and non-abelian Chern-Simons terms. We see, too, that an alternative index defined in terms of commutation properties of the fields, while less fundamental, does naturally produce this integer value. The equality of I_A and I_Z can be a useful calculational tool; since Ω eigenvalues, unlike J eigenvalues, cannot vary continuously even in $2+1$ dimensions, the index I_Z is easier to calculate reliably.

It would be most intriguing to find an example where an integral index I_A arises from a truly exotic combination of vacuum statistics, not simply from pairwise cancellations, although the possibility of this is severely constrained by the equality of I_A and I_Z. In addition, combining index calculations with an analysis of the spectrum based on other techniques might yield interesting relationships among the states in a theory with anyons in a relatively easy and efficient manner, potentially even yielding an alternative avenue for identifying the appearance of anyon behavior itself. Even more speculatively, perhaps one could find a condensed matter system which is effectively planar and exhibits a phenomenological supersymmetry to which our Witten index results could be applied.

Acknowledgments

I thank Al Shapere for discussions. I also thank the Theory Group at NIKHEF for its hospitality. This work was supported in part by NSF Grant. No. PHY-9509991.

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